

# Relating Infinite Set Theory to Other Branches of Mathematics

**Roads to Infinity: The Mathematics of Truth and Proof.** By John Stillwell, AK Peters, Natick, Massachusetts, 2010, 250 pages, \$39.00.

The infinite, wrote Jorge Luis Borges, is a concept that “corrupts and confuses the others.” Certainly, the theory of large infinite sets has always seemed to me the most remote area of mathematical logic—remote both from the other areas of logic and from computer science, which is where I work. Computation theory, recursion theory, and automata theory are, of course, our bread and butter here in CS, and model theory and proof theory are well-worn tools. But infinite set theory turns up only rarely; there is occasionally a proof by transfinite induction, or something of the kind. I was therefore particularly interested to read John Stillwell’s new book, *Roads to Infinity: The Mathematics of Truth and Proof*, which gives an account of the development of infinite set theory, and its relation to proof theory, computability and decidability theory, and other branches of mathematics. Countering Borges, Stillwell demonstrates that the theory of infinite sets enriches and clarifies many others.

## BOOK REVIEW

By Ernest Davis

John Stillwell is a mathematician and math historian at the University of San Francisco. He is the author of numerous math books, both popular and technical; he won the Chauvenet Prize in 2005 for his article “The Story of the 120-Cell” and the Association of Jesuit Colleges and Universities book prize in 2009 for his book *Yearning for the Impossible: The Surprising Truths of Mathematics*.

I would classify *Roads to Infinity* as semi-popular—it is certainly neither a technical book nor a textbook—but it is one of the more challenging semi-popular math books that I have seen. Stillwell writes in the introduction that it requires only high-school math. Strictly speaking, this is true: The book is very much self-contained, uses little of even the high-school math curriculum, and contains no proofs longer than about a page. In practice, though, a reader would not get far without both that intangible quality “mathematical sophistication” and, more specifically, comfortable familiarity with  $\aleph_0$ ,  $2^{\aleph_0}$  and with the notion of a proof as a symbolic structure. Chapters 1 and 2 are relaxed and discursive, but the writing in the later chapters, though clear, is very compact. In some ways, the book would be best suited as supplemental light reading for an introductory course on mathematical logic, to help the students see the forest rather than the trees. If you’re looking for a readable book on logic for your bright, math-oriented, teenaged nephew or niece, Hofstadter’s *Gödel, Escher, Bach* is still the place to start; if a tougher challenge is called for, you can follow up with *Roads to Infinity*.

Stillwell begins in familiar territory, with countable sets, uncountable sets, and Cantor’s proof that the two are unequal. Each chapter concludes with a “historical background” section that, contrary to the usual practice, sometimes contains important mathematical concepts not included elsewhere. Presented as historical background, for example, are the Zermelo–Fraenkel axioms (chapter 1), and the axiom of choice and the theory of measurable sets (chapter 2). (Stillwell’s suggestion in the preface that a reader might choose to skip the historical background should not be followed.)

Chapter 2 presents the theory of the ordinals and well-ordered sets. It concludes with two remarkable examples of algorithms—one a number-theoretic algorithm of Goldstein, the other a graph-theoretic algorithm of Kirby and Paris—whose termination proofs depend on showing that the state corresponds to a descending sequence of infinite ordinals. I had never seen this, and would not have thought it possible.

Leaving infinite sets for a while, Stillwell turns in chapters 3 and 4 to computation and undecidability theory. These chapters are adequate but the weakest part of the book; several other books present this material better. The most interesting aspect of these chapters is the historical discussion of Emil Post’s early work. In 1921, ten years before Gödel’s incompleteness theorem, Post proved, without publishing the work, that any sound formal system contains true unprovable statements. Nevertheless, Stillwell’s description of Post as a “neglected figure” seems to me an overstatement. Aside from Russell, Gödel, and Turing, each known to the educated public for one reason or another, I don’t suppose that Post is much less well known than other 20th-century logicians.

Chapters 5, 6, and 7 are the high point of the book. The material here was almost all new to me. Chapter 5 discusses Peano arithmetic,  $\omega$ -induction, and  $\varepsilon_0$ -induction, thus demonstrating the relation between the theory of ordinals and the theory of proof. It also discusses the issues involved in proving the consistency of arithmetic. Chapter 6 offers examples of “natural” sentences that are true but unprovable in Peano arithmetic. Stillwell describes some interesting examples from Ramsey theory and graph theory, such as Kruskal’s theorem:

For any infinite sequence of trees  $T_1, T_2, \dots$  there exists  $i < j$  such that  $T_i$  embeds in  $T_j$ .

As Stillwell mentions, however, none of the known number-theoretic theorems unprovable in Peano arithmetic have been found or even conjectured by number theorists; all have been found by logicians. “Natural” is a matter of taste.

Chapter 7 returns to infinite set theory, the axiomatization of infinite sets, and descriptive set theory. Remarkably, the theory of large infinite sets can have consequences and give insights into the finite world. Stillwell describes work by Dehornoy that used results from large cardinal theory to prove an important result in braid theory. There is also the following extraordinary, general result\*:

Let  $\phi$  be an axiom that asserts the existence of an infinite cardinal  $C$  at some level of the hierarchy of infinities. Then there exists an integer polynomial  $p(x_1, \dots, x_n)$  such that

\*I am grateful to Martin Davis for discussions on this point.

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- a. One can prove from ZFC +  $\phi$  that the Diophantine equation  $p(x_1 \dots x_n) = 0$  has no solutions.
- b. This cannot be proven from ZFC +  $\psi$ , where  $\psi$  is any axiom asserting the existence of a cardinal lower in the hierarchy than  $C$ .

The discussion in the last three chapters is necessarily largely expository, as all but a few of the proofs are far too long and difficult to include or even sketch. But the presentations of the concepts and their interconnections, both mathematical and historical, are impressively clear, and quite fascinating.

The book has a few gaps. Despite the extensive discussion of measure theory, there is no mention of the Cantor set, and no clear statement of the existence of uncountable sets of measure zero, which is critical to Lebesgue measure. The book's presentation of the ZF axioms, before predicate logic is defined, is necessarily informal; it would have been worthwhile, after presenting predicate logic, to go back and show how they are axiomatized. There is very little discussion of the axiom of replacement, the most difficult of the ZF axioms. An example explaining it would have been useful, along with a detailed discussion of how a naive statement of the comprehension axiom gives rise to Russell's paradox, which the axiom of replacement avoids while still constructing all the sets that one actually needs.

Stillwell presents a proof by Harnack (1885) that the real interval  $[0,1]$  is uncountable, using the fact that any countable subset of  $[0,1]$  can be covered by a collection of intervals of arbitrarily small total measure. However, Stillwell omits the critical and non-trivial step that the entire interval  $[0,1]$  *can't* be covered by such a collection (i.e., that this is a countably additive measure). If I were discussing undecidability, I would mention, by way of contrast, some of the more remarkable decidability results, such as Tarski's proof of the decidability of the first-order theory of real arithmetic. I would not discuss Cook's proof that satisfiability is NP-complete without mentioning Karp's result that a lot of other fundamental algorithmic problems are also NP-complete; but that, perhaps, reflects the difference between the mathematician and the computer scientist.

The few typos I noticed were mostly unimportant; the omission of "+1" from Euclid's proof of the infinitude of primes, however, is the kind of thing that makes an author want to recall the entire print run and correct every copy by hand.

Reading *Roads to Infinity* clarified for me many things of which I previously had only a muddled idea, and taught me many things of which I had no idea. I am glad to have read it, and grateful to the author. But I can't say that it moved me with any burning desire to learn more about large infinite sets. Inaccessible cardinals still seem remote.

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