

Nonlinear Ocean-Wave Interactions on Flat Beaches

By *M.J. Ablowitz and D.E. Baldwin*

People have always been fascinated by waves, particularly water and ocean waves. The mathematical study of water waves goes back to the origins of differential equations. While linear equations are often good models for small-amplitude waves, nonlinear equations are needed for larger amplitudes. We have seen and photographed interacting nonlinear waves that occur daily at two relatively flat beaches; a well-known nonlinear wave equation has solutions that are remarkably similar to what we observed.

Even Newton (1642–1727) was interested in providing a mathematical description of water waves, but many years would pass before this was feasible. In 1757 Euler derived the inviscid equations of fluid dynamics. Soon afterward, Laplace and Lagrange found linear approximations to the water-wave equations. In 1816 Cauchy’s study of the linear initial-value problem of water waves won a prize from the French Academy of Sciences. This work, an early application of Fourier analysis, was not well understood at the time. But in general, water-wave dynamics satisfy nonlinear equations because the wave amplitudes are not sufficiently small.

In 1847 Stokes derived the correct nonlinear boundary conditions on the water’s free surface and used it to show that the speed of a traveling wave in deep water depends on its amplitude. In the 1870s, understanding that the nonlinear water-wave equations are simplified when the water is shallow or the waves are long, Boussinesq derived (1+1)-dimensional equations (one space and one time dimension); he found a solitary wave solution that is localized and nonperiodic. In 1895 Korteweg and his student de Vries followed Boussinesq’s pioneering path and derived a unidirectional (1+1)-dimensional nonlinear equation for shallow water, usually called the Korteweg–de Vries (KdV) equation. They also found special periodic solutions, which they called cnoidal waves, that can be written in terms of Jacobian elliptic functions. The cnoidal wave, in a special limit, becomes a solitary wave. A solitary wave had been observed in 1834 by Russell, a naval engineer; he found that the wave’s speed depends on its amplitude, which agrees with the KdV equation’s solitary wave.

Between 1895 and 1960, most applications of the KdV equation involved water waves. But in the 1960s mathematicians found that the KdV equation is universal: It arises in wave problems with weak dispersion and weak quadratic nonlinearity. Besides water waves, the KdV equation arises in stratified fluids, plasma physics, elasticity, and lattice dynamics, among other settings. It was lattice dynamics that motivated Kruskal and Zabusky in 1965 to carry out a numerical study of the KdV equation. They discovered that these KdV solitary waves have special interaction properties: Their amplitudes before and after interaction are preserved, but there is a phase shift. They called these special solitary waves solitons. Soon afterward, in 1967, Gardner, Greene, Kruskal, and Miura developed a method—later named the inverse scattering transform (IST) method—for finding the solution. They also found a spectral interpretation for solitons. Their work spurred great interest, and many researchers made important contributions.

Equations solvable with the IST method, like the KdV equation, are often called integrable. In 1970 Kadomtsev and Petviashvili (KP) found a multidimensional (two-space, one-time) generalization of the KdV equation; it is also integrable and can be derived from the water-wave equations in shallow water with surface tension included. Like the KdV equation, it has soliton solutions that can be written explicitly. The simplest is a plane-wave solution, which is essentially one-dimensional and satisfies the KdV equation.

The well-known two-soliton solutions, first found in the 1970s, are more interesting; surprisingly, similar interactions are visible on a daily basis on relatively flat beaches. It is useful to write the two-soliton solution of the KP equation, with small surface tension, with a phase-shift parameter that we label e^ϕ . We concentrate on four cases: e^ϕ order one, e^ϕ large, e^ϕ zero, and e^ϕ small. Remarkably, we have seen each of these types at the beach; we call them short-stem X-, long-stem X-, Y-, and H-type interactions, respectively.

Before our recent observations, there was only one known photograph—of a long-stem X-type interaction, taken on the Oregon coast in the 1970s (see [1], page 291). MJA saw and photographed short- and long-stem X-type and Y-type interactions in Nuevo Vallarta, Mexico; he also occasionally saw and photographed more complex multisoliton interactions. Motivated by this and the KP equation’s analytic solutions, DEB traveled to Venice Beach, California, where he saw and photographed not only interactions of the types seen by MJA, but also H-type interactions. We observed these soliton interactions daily on relatively flat beaches, in shallow water, within about two hours of low tide. Being near a jetty helps the development of cross-waves but is not necessary if there is a good crosswind. We have seen mainly two-soliton interactions but occasionally have spotted more complex soliton interactions as well. Additional details and photos can be found in our paper [2], and we have also posted photos and videos on our websites (<http://www.markablowitz.com/line-solitons> and <http://www.douglasbaldwin.com/nl-waves.html>).

Along with the phase shift, some of these distinctive nonlinear interactions are explained in part by the stem height: Not just the sum of the wave heights away from the interaction, it can be considerably higher. This can be important in descriptions of tsunami propagation, which in certain cases can be modeled with the KP equation. Indeed, satellite images reveal local X- and Y-type interactions for the 2011 Japanese earthquake-induced tsunami. This made the effects of the tsunami even worse. Because the Japanese tsunami was close to shore,

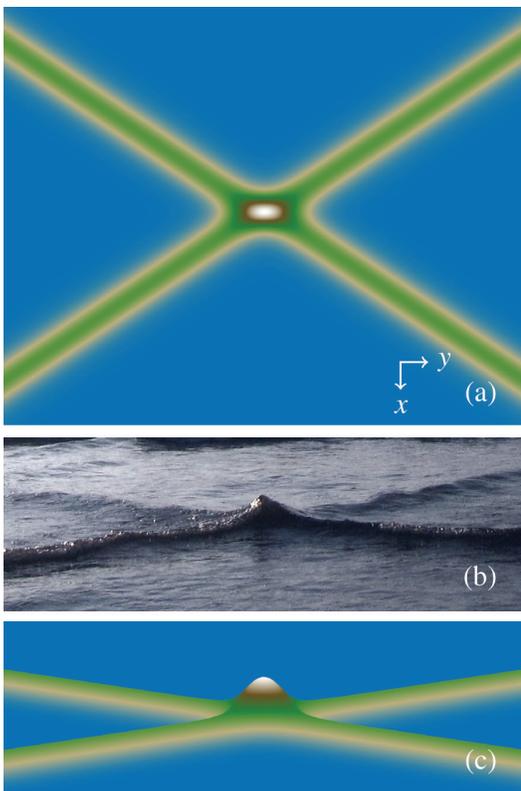


Figure 1. Short-stem X-type interaction; see also [2]. (a) Contour plot of an analytical line-soliton interaction solution of the KP equation (here $e^\phi \approx 2:3$). (b) Photograph taken in Mexico, December 31, 2011. (c) 3D plot of the solution shown in (a).

nonlinearity did not have time to amplify the stem height; other tsunamis might occur well away from shore, in which case nonlinear effects could become important. In such cases X- and Y-type tsunami interactions could be extremely destructive.

References

[1] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

[2] M.J. Ablowitz and D.E. Baldwin, *Non-linear shallow ocean-wave soliton interactions on flat beaches*, Phys. Rev. E, 86 (2012), 036305.

M.J. Ablowitz is a professor of applied mathematics at the University of Colorado, Boulder, from which D.E. Baldwin recently received a PhD.

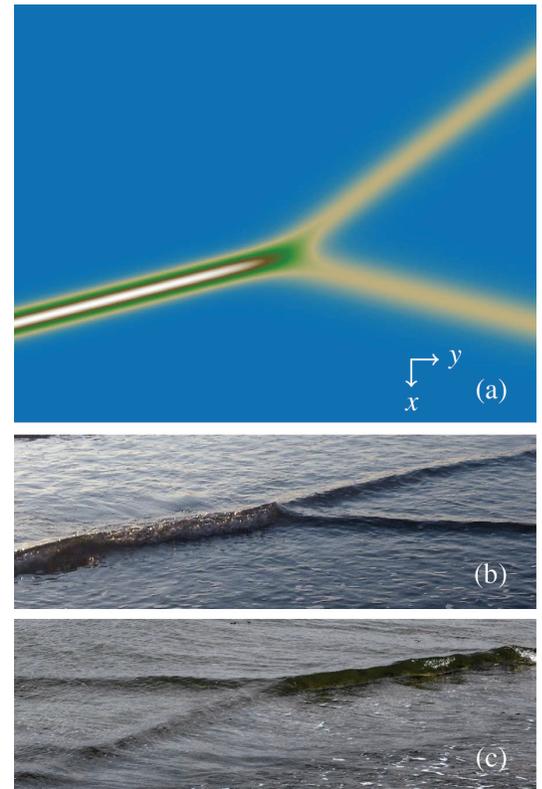


Figure 2. Contour plot (a) and photographs (b), (c) of a Y-type interaction ($e^0 = 0$); see also [2]. (b) Taken in Mexico, January 6, 2010. (c) Taken in California, May 3, 2012.