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Developing a Computation-Friendly Mathematical Foundation for Spline Functions

By Ronald DeVore and Amos Ron

*Polynomials are wonderful even after they are cut into pieces, but the cutting must be done with care. One way of doing the cutting leads to the so-called spline functions.**

Iso Schoenberg, "the father of splines," penned these prophetic words in 1964. That same year, Carl de Boor entered graduate school at the University of Michigan. De Boor's name was mentioned only briefly in Schoenberg's paper, in a "Note added in proof." In fact, when informed by de Boor a few years later of his newly found recurrence relation for B-splines, Schoenberg was initially skeptical: While linear combinations of non-smooth functions could, accidentally, produce a function of greater smoothness, one should not expect such miracles to happen in a consistent way! Schoenberg's skepticism, however, was short-lived: In 1972, on his initiative, the University of Wisconsin hired de Boor at the rank of full professor.

Univariate spline functions are piecewise polynomials that are carefully joined at their breakpoints. Although spline functions date back to work of Euler and Bernoulli, it was Iso Schoenberg who began to study them in earnest. Spurred by their potential application to data fitting, Schoenberg and his co-authors proved many fundamental properties of spline functions. With the advent of high-speed computers, it became clear that spline functions could provide efficient representations of curves and surfaces. But making that happen would require fast algorithms for evaluating splines and for generating efficient approximations.

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Carl de Boor understood the potential of spline functions in the early 60s, during his time at the General Motors Research Center, where splines were already being employed in automotive design. Together with other notable mathematicians, such as Garrett Birkhoff and John Rice, de Boor embarked on an ambitious program to develop a mathematical foundation for spline functions that would be friendly to computation. The cornerstone of this development would be the B-splines: splines with minimal support.

In his early work, de Boor dealt with bicubic spline interpolation and approximation by splines with variable knots. But his focus soon shifted to understanding how B-splines could be used to represent general spline functions. In the early 70s, he showed, among other things, that the smoother, higher-degree B-spline can be written as a combination of two lower-degree, and hence lower-smoothness, B-splines, with coefficients that are positive linear polynomials on the spline's support. With George Fix, he later found a remarkable formula for the coefficients of the B-splines in a spline representation. Using that formula, de Boor went on to prove what might be considered the most counterintuitive property of splines: The B-spline basis has a condition number that is bounded independently of the distribution of the knot sequence. The de Boor–Fix formula has proved to be the right vehicle for establishing many important theoretical properties of spline functions.

Spline approximations to curves and functions are typically constructed via projection onto spline spaces. Understanding the approximation properties of such projections requires bounds for their norms in appropriate topologies. By analyzing the matrices that represent these projectors, de Boor established deep properties of spline projectors. One of his famous conjectures, in the mid-1970s, was that, for any knot sequence and any polynomial degree, the least-squares projector onto spline functions with this knot sequence is bounded in the uniform norm, independently of the knot sequence. Alexei Shadrin recently settled this conjecture in the affirmative.

The initial development of spline functions was inherently univariate. Although multidimensional splines are easily constructed via tensor products, de Boor proposed an approach to multidimensional construction that would dramatically change the spline landscape. As pointed out by Curry and Schoenberg, univariate B-splines admit a beautiful geometric interpretation: The interpretation starts by mapping a properly chosen high-dimensional simplex linearly to 1D, and then shows that the value of the B-spline at a point x is represented as the cross-sectional volume of the points in the simplex that are mapped to x. In 1976, de Boor noted that this interpretation could be used as the *definition* of compactly supported piecewise-polynomial multivariate B-splines. The usefulness of such new splines, however, would require, among other things, a stable recurrence relation for those splines.

A few years later, C.A. Micchelli and, independently, W. Dahmen established the sought-after recurrence relations. Various important properties of these *simplicial* B-splines were thoroughly established by Dahmen and Micchelli. But combinatorial and topological difficulties soon muddied the simplicial B-spline waters. The time was ripe for the advent of one of the most dramatic successes of multivariate splines: *box splines*.

Box splines are compactly supported multivariate piecewise-polynomial functions that are obtained by replacing the aforementioned simplex by a cube. Suitable box spline spaces are then defined as the span of *integer* translates of the box spline. Initially introduced as an approximation tool, box splines have led researchers to myriad intriguing research problems in approximation theory, in combinatorics, in diophantine equations, in commutative algebra, and in several other fields. As one example, Rong-Qing Jia, a former student of de Boor's, used box spline theory to settle (in the late 80s) a conjecture of Richard Stanley concerning the number of solutions for the symmetric magic square problem. Box splines also give a simple multiscale representation system for general functions that can be viewed as a precursor to wavelet constructions. Moreover, being *refinable*, box splines can be, and are, used in the construction of multivariate wavelets. Box splines have also been investigated as a tool for *subdividing* surfaces. The surface subdivision algorithm most commonly used today, the Loop scheme, is box-spline-based.

Box splines made their debut in 1983, in a joint paper of Carl de Boor and Ronald DeVore. The basic theory had been developed in the early 80s by de Boor and his younger Wisconsin colleague Klaus Höllig and, in parallel, by Dahmen and Micchelli. Two of the notable early results on box splines were the de Boor–Höllig characterization of the polynomials in the box spline spaces in terms of annihilating differential operators, and the Dahmen–Micchelli dimension formula for that polynomial space in terms of the number of tiles used for covering the box spline's support.

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In 1988, Amos Ron succeeded Höllig at Wisconsin. Building on Ron's more general, and generically simpler, *exponential* box splines, de Boor and Ron jointly developed a theory for computing dimensions of polynomial spaces based on perturbing the generators of polynomial ideals. As is often the case in mathematics, de Boor and Ron observed that the theory led to an unexpected spin-off: an association of each finite pointset in $\mathbf{R}^d$ with a polynomial space, *the least space of the multivariate polynomial interpolation problem*, on which interpolation is correct, i.e., uniquely solvable. A cornerstone of that theory is the connection between algebraic varieties that may contain the pointset and differential operators that annihilate the least polynomial space. If, for example, the pointset lies on a sphere, then the least space must be annihilated by the Laplacian and therefore must consist of harmonic polynomials.

In the early 90s, de Boor and Ron laid the foundation for what are known today as *shift-invariant spaces*, i.e., multivariate function spaces generated by the integer translates of one or finitely many "generators." Departing from the then-prevailing quasi-interpolation approaches that were largely confined to piecewise polynomials, de Boor and Ron used Fourier analysis techniques to develop suitable approximation schemes. This theoretical work culminated in a series of papers in which de Boor, DeVore, and Ron characterized the approximation properties of shift-invariant spaces in complete generality, and made important connections to wavelet constructions.

De Boor's contributions to splines, to approximation theory, to scientific computing, to mathematics, and to science have not gone unnoticed. In addition to the National Medal of Science he received this year, he holds two named professorships at the University of Wisconsin, and has received honorary degrees from Purdue and the Technion. He has been elected to numerous academic societies in the U.S. and in Europe, including the U.S. National Academies of Sciences and of Engineering. SIAM honored de Boor in 1996 by naming him the John von Neumann lecturer.

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